# 42nd International Mathematical Olympiad

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## Problems

*Each problem is worth seven points.*

### Problem 1

Let $ABC$ be an acute-angled triangle with circumcentre $O$. Let $P$ on $BC$ be the foot of the altitude from $A$.

Suppose that $\angle BCA \geq \angle ABC + 30^\circ$.

Prove that $\angle CAB + \angle COP < 90^\circ$.

### Problem 2

Prove that

$$\frac{a}{\sqrt{a^2 + 8bc}} + \frac{b}{\sqrt{b^2 + 8ca}} + \frac{c}{\sqrt{c^2 + 8ab}} \geq 1$$

for all positive real numbers $a$, $b$ and $c$.

### Problem 3

Twenty-one girls and twenty-one boys took part in a mathematical contest.

- Each contestant solved at most six problems.
- For each girl and each boy, at least one problem was solved by both of them.

Prove that there was a problem that was solved by at least three girls and at least three boys.

### Problem 4

Let $n$ be an odd integer greater than 1, and let $k_1, k_2, \ldots, k_n$ be given integers. For each of the $n!$ permutations $a = (a_1, a_2, \ldots, a_n)$ of $1, 2, \ldots, n$, let

$$S(a) = \sum_{i=1}^{n} k_i a_i.$$

Prove that there are two permutations $b$ and $c$, $b \neq c$, such that $n!$ is a divisor of $S(b) - S(c)$.

## Problem 5

In a triangle $ABC$, let $AP$ bisect $\angle BAC$, with $P$ on $BC$, and let $BQ$ bisect $\angle ABC$, with $Q$ on $CA$.

It is known that $\angle BAC = 60^\circ$ and that $AB + BP = AQ + QB$.

What are the possible angles of triangle $ABC$?

## Problem 6

Let $a, b, c, d$ be integers with $a > b > c > d > 0$. Suppose that

$$a\,c + b\,d = (b + d + a - c)\,(b + d - a + c).$$

Prove that $a\,b + c\,d$ is not prime.

# Problems with Solutions

## Problem 1

Let $ABC$ be an acute-angled triangle with circumcentre $O$. Let $P$ on $BC$ be the foot of the altitude from $A$.

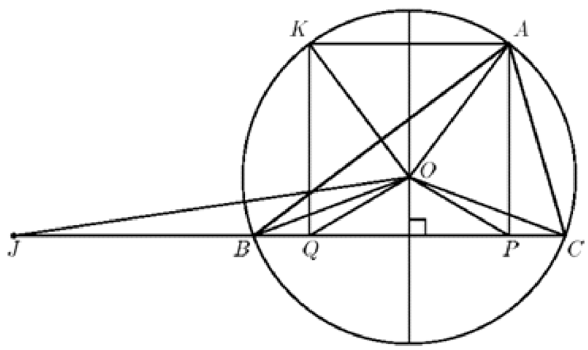
Suppose that $\angle BCA \geq \angle ABC + 30^\circ$.

Prove that $\angle CAB + \angle COP < 90^\circ$.

### Solution

- Solution 1

Let $\alpha = \angle CAB$, $\beta = \angle ABC$, $\gamma = \angle BCA$, and $\delta = \angle COP$. Let $K$ and $Q$ be the reflections of $A$ and $P$, respectively, across the perpendicular bisector of $BC$. Let $R$ denote the circumradius of $\triangle ABC$. Then $OA = OB = OC = OK = R$. Furthermore, we have $QP = KA$ because $KQPA$ is a rectangle. Now note that $\angle AOK = \angle AOB - \angle KOB = \angle AOB - \angle AOC = 2\gamma - 2\beta \geq 60^\circ$.




It follows from this and from $OA = OK = R$ that $KA \geq R$ and $QP \geq R$. Therefore, using the Triangle Inequality, we have $OP + R = OQ + OC > QC = QP + PC \geq R + PC$. It follows that $OP > PC$, and hence in $\triangle COP$, $\angle PCO > \delta$. Now since $\alpha = \frac{1}{2}\angle BOC = \frac{1}{2}(180^\circ - 2\angle PCO) = 90^\circ - \angle PCO$, it indeed follows that $\alpha + \delta < 90^\circ$.

- Solution 2

As in the previous solution, it is enough to show that $OP > PC$. To this end, recall that by the (Extended) Law of Sines, $AB = 2R\sin\gamma$ and $AC = 2R\sin\beta$. Therefore, we have

$$BP - PC = \mathrm{AB}\cos\beta - A\,C\cos\gamma = 2\,R(\sin\gamma\cos\beta - \sin\beta\cos\gamma) = 2\,R\sin(\gamma - \beta).$$

It follows from this and from

$$30^\circ \leq \gamma - \beta < \gamma < 90^\circ$$

that $BP - PC \geq R$. Therefore, we obtain that $R + OP = BO + OP > BP \geq R + PC$, from which $OP > OC$, as desired.

■ Solution 3

We first show that $R^2 > CP \cdot CB$. To this end, since $CB = 2R\sin\alpha$ and $CP = AC\cos\gamma = 2R\sin\beta\cos\gamma$, it suffices to show that $\frac{1}{4} > \sin\alpha\sin\beta\cos\gamma$. We note that $1 > \sin\alpha = \sin(\gamma + \beta) = \sin\gamma\cos\beta + \sin\beta\cos\gamma$ and $\frac{1}{2} \leq \sin(\gamma - \beta) = \sin\gamma\cos\beta - \sin\beta\cos\gamma$ since $30^\circ \leq \gamma - \beta < 90^\circ$. It follows that $\frac{1}{4} > \sin\beta\cos\gamma$ and that $\frac{1}{4} > \sin\alpha\sin\beta\cos\gamma$.

Now we choose a point $J$ on $BC$ so that $CJ \cdot CP = R^2$. It follows from this and from $R^2 > CP \cdot CB$ that $CJ > CB$, so that $\angle OBC > \angle OJC$. Since $OC / CJ = PC / CO$ and $\angle JCO = \angle OCP$, we have $\triangle JCO \cong \triangle OCP$ and $\angle OJC = \angle POC = \delta$. It follows that $\delta < \angle OBC = 90^\circ - \alpha$ or $\alpha + \delta < 90^\circ$.

■ Solution 4

On the one hand, as in the third solution, we have $R^2 > CP \cdot CB$. On the other hand, the power of $P$ with respect to the circumcircle of $\triangle ABC$ is $BP \cdot PC = R^2 - OP^2$. From these two equations we find that

$$OP^2 = R^2 - BP \cdot PC > PC \cdot CB - BP \cdot PC = PC^2,$$

from which $OP > PC$. Therefore, as in the first solution, we conclude that $\alpha + \delta < 90^\circ$.

## Problem 2

Prove that

$$\frac{a}{\sqrt{a^2 + 8\,b\,c}} + \frac{b}{\sqrt{b^2 + 8\,c\,a}} + \frac{c}{\sqrt{c^2 + 8\,a\,b}} \geq 1$$

for all positive real numbers $a$, $b$ and $c$.

### Solution

First we shall prove that

$$\frac{a}{\sqrt{a^2 + 8\,b\,c}} \geq \frac{a^{\frac{4}{3}}}{a^{\frac{4}{3}} + b^{\frac{4}{3}} + c^{\frac{4}{3}}},$$

or equivalently, that

$$\left(a^{\frac{4}{3}} + b^{\frac{4}{3}} + c^{\frac{4}{3}}\right)^2 \geq a^{\frac{2}{3}}\,(a^2 + 8\,b\,c).$$

The AM-GM inequality yields

$$\begin{aligned} \left(a^{\frac{4}{3}} + b^{\frac{4}{3}} + c^{\frac{4}{3}}\right)^2 - \left(a^{\frac{4}{3}}\right)^2 &= \left(b^{\frac{4}{3}} + c^{\frac{4}{3}}\right)\left(a^{\frac{4}{3}} + a^{\frac{4}{3}} + b^{\frac{4}{3}} + c^{\frac{4}{3}}\right) \\ &\geq 2\,b^{\frac{2}{3}}\,c^{\frac{2}{3}} \cdot 4\,a^{\frac{2}{3}}\,b^{\frac{1}{3}}\,c^{\frac{1}{3}} \\ &= 8\,a^{\frac{2}{3}}\,b\,c. \end{aligned}$$

Thus

$$\left(a^{\frac{4}{3}} + b^{\frac{4}{3}} + c^{\frac{4}{3}}\right)^2 \geq \left(a^{\frac{4}{3}}\right)^2 + 8\,a^{\frac{2}{3}}\,b\,c$$
$$= a^{\frac{2}{3}}(a^2 + 8\,b\,c),$$

so

$$\frac{a}{\sqrt{a^2 + 8\,b\,c}} \geq \frac{a^{\frac{4}{3}}}{a^{\frac{4}{3}} + b^{\frac{4}{3}} + c^{\frac{4}{3}}}.$$

Similarly, we have

$$\frac{b}{\sqrt{b^2+8\,c\,a}} \geq \frac{b^{\frac{4}{3}}}{a^{\frac{4}{3}}+b^{\frac{4}{3}}+c^{\frac{4}{3}}} \quad \text{and}$$
$$\frac{c}{\sqrt{c^2+8\,a\,b}} \geq \frac{c^{\frac{4}{3}}}{a^{\frac{4}{3}}+b^{\frac{4}{3}}+c^{\frac{4}{3}}}.$$

Adding these three inequalities yields

$$\frac{a}{\sqrt{a^2 + 8\,b\,c}} + \frac{b}{\sqrt{b^2 + 8\,c\,a}} + \frac{c}{\sqrt{c^2 + 8\,a\,b}} \geq 1.$$

*Comment*. It can be shown that for any $a, b, c > 0$ and $\lambda \geq 8$, the following inequality holds:

$$\frac{a}{\sqrt{a^2 + \lambda\,b\,c}} + \frac{b}{\sqrt{b^2 + \lambda\,c\,a}} + \frac{c}{\sqrt{c^2 + \lambda\,a\,b}} \geq \frac{3}{\sqrt{1 + \lambda}}.$$

## Problem 3

Twenty-one girls and twenty-one boys took part in a mathematical contest.

- Each contestant solved at most six problems.
- For each girl and each boy, at least one problem was solved by both of them.

Prove that there was a problem that was solved by at least three girls and at least three boys.

### Solution

- Solution 1

We introduce the following symbols: $G$ is the set of girls at the competition and $B$ is the set of boys, $P$ is the set of problems, $P(g)$ is the set of problems solved by $g \in G$, and $P(b)$ is the set of problems solved by $b \in B$. Finally, $G(p)$ is the set of girls that solve $p \in P$ and $B(p)$ is the set of boys that solve $p$. In terms of this notation, we have that for all $g \in G$ and $b \in B$,

$$(a)\ \ |P(g)| \leq 6,\ \ |P(b)| \leq 6, \quad (b)\ \ P(g) \cap P(b) \neq \emptyset.$$

We wish to prove that some $p \in P$ satisfies $|G(p)| \geq 3$ and $|B(p)| \geq 3$. To do this, we shall assume the contrary and reach a contradiction by counting (two ways) all ordered triples $(p, q, r)$ such that $p \in P(g) \cap P(b)$. With $T = \{(p, g, b) :\ p \in P(g) \cap P(b)\}$, condition (b) yields

$$|T| = \sum_{g\in G}\sum_{b\in B} |P(g) \cap P(b)| \geq |G| \cdot |B| = 21^2. \tag{1}$$

Assume that no $p \in P$ satisfies $|G(p)| \geq 3$ and $|B(p)| \geq 3$. We begin by noting that

$$\sum_{p\in P} |G(p)| = \sum_{g\in G} |P(g)| \leq 6|G| \quad \text{and} \quad \sum_{p\in P} |B(p)| \leq 6|B|. \tag{2}$$

(*Note*. The equality in (2) is obtained by a standard double-counting technique: Let $\chi(g, p) = 1$ if $g$ solves $p$ and $\chi(g, p) = 0$ otherwise, and interchange the orders of summation in $\sum_{p\in P}\sum_{g\in G}\chi(g, p)$.) Let

$$P_+ = \{p \in P : |G(p)| \geq 3\},$$
$$P_- = \{p \in P : |G(p)| \leq 2\}.$$

**Claim.** $\sum_{p\in P_-} |G(p)| \geq |G|$; thus $\sum_{p\in P_+} |G(p)| \leq 5|G|$. Also $\sum_{p\in P_+} |B(p)| \geq |B|$; thus $\sum_{p\in P_-} |B(p)| \leq 5|B|$.

*Proof*. Let $g \in G$ be arbitrary. By the Pigeonhole Principle, conditions (a) and (b) imply that $g$ solves some problem $p$ that is solved by at least $\lceil 21/6 \rceil = 4$ boys. By assumption, $|B(p)| \geq 4$ implies that $p \in P_-$, so every girl solves at least one problem in $P_-$. Thus

$$\sum_{p\in P_-} |G(p)| \geq |G|. \tag{3}$$

In view of (2) and (3) we have

$$\sum_{p\in P_+} |G(p)| = \sum_{p\in P} |G(p)| - \sum_{p\in P_-} |G(p)| \leq 5|G|.$$

Also, each boy solves a problem that is solved by at least four girls, so each boy solves a problem $p \in P_+$. Thus $\sum_{p\in P_+} |B(p)| \geq |B|$, and the calculation proceeds as before using (2). □

Using the claim just established, we find

$$\begin{aligned} |T| &= \textstyle\sum_{p\in P} |G(p)| \cdot |B(p)| \\ &= \textstyle\sum_{p\in P_+} |G(p)| \cdot |B(p)| + \sum_{p\in P_-} |G(p)| \cdot |B(p)| \\ &\leq 2\textstyle\sum_{p\in P_+} |G(p)| + 2\sum_{p\in P_-} |B(p)| \\ &\leq 10|G| + 10|B| = 20\cdot 21. \end{aligned}$$

This contradicts (1), so the proof is complete.

■ Solution 2

Let us use some of the notation given in the first solution. Suppose that for every $p \in P$ either $|G(p)| \leq 2$ or $|B(p)| \leq 2$. For each $p \in P$, color $p$ red if $|G(p)| \leq 2$ and otherwise color it black. In this way, if $p$ is red then $|G(p)| \leq 2$ and if $p$ is black then $|B(p)| \leq 2$. Consider a chessboard with 21 rows, each representing one of the girls, and 21 columns, each representing one of the boys. For each $g \in G$ and $b \in B$, color the square corresponding to $(g, b)$ as follows: pick $p \in P(g) \bigcap P(b)$ and assign $p$'s color to that square. (By condition (b), there is always an available choice.) By the Pigeonhole Principle, one of the two colors is assigned to at least $\lceil 441/2 \rceil = 221$ squares, and thus some row has at least $\lceil 221/21 \rceil = 11$ black squares or some column has at least 11 red squares.

Suppose the row corresponding to $g \in G$ has at least 11 black squares. Then for each of 11 squares, the black problem that was chosen in assigning the color was solved by at most 2 boys. Thus we account for at least $\lceil 11/2 \rceil = 6$ distinct problems solved by $g$. In view of condition (a), $g$ solves only these problems. But then at most 12 boys solve a problem also solved by $g$, in violation of condition (b).

In exactly the same way, a contradiction is reached if we suppose that some column has at least 11 red squares. Hence some $p \in P$ satisfies $|G(p)| \geq 3$ and $|B(p)| \geq 3$.

## Problem 4

Let $n$ be an odd integer greater than 1, and let $k_1, k_2, \ldots, k_n$ be given integers. For each of the $n!$ permutations $a = (a_1, a_2, \ldots, a_n)$ of $1, 2, \ldots, n$, let

$$S(a) = \sum_{i=1}^{n} k_i \, a_i .$$

Prove that there are two permutations $b$ and $c$, $b \neq c$, such that $n!$ is a divisor of $S(b) - S(c)$.

### Solution

Let $\sum S(a)$ be the sum of $S(a)$ over all $n!$ permutations $a = (a_1, a_2, \ldots, a_n)$. We compute $\sum S(a) \bmod n!$ two ways, one of which depends on the desired conclusion being false, and reach a contradiction when $n$ is odd.

*First way*. In $\sum S(a)$, $k_1$ is multiplied by each $i \in \{1, \ldots, n\}$ a total of $(n-1)!$ times, once for each permutation of $\{1, \ldots, n\}$ in which $a_1 = i$. Thus the coefficient of $k_1$ in $\sum S(a)$ is

$$(n-1)!\,(1 + 2 + \cdots + n) = (n+1)!/2.$$

The same is true for all $k_i$, so

$$\sum S(a) = \frac{(n+1)!}{2} \sum_{i=1}^{n} k_i . \tag{1}$$

*Second way*. If $n!$ is not a divisor of $S(b) - S(c)$ for any $b \neq c$, then each $S(a)$ must have a different remainder mod $n!$. Since there are $n!$ permutations, these remainders must be precisely the numbers $0, 1, 2, \ldots, n! - 1$. Thus

$$\sum S(a) \equiv \frac{(n! - 1)\, n!}{2} \bmod n!. \tag{2}$$

Combining (1) and (2), we get

$$\frac{(n+1)!}{2} \sum_{i=1}^{n} k_i \equiv \frac{(n! - 1)\, n!}{2} \bmod n!. \tag{3}$$

Now, for $n$ odd, the left side of (3) is congruent to 0 modulo $n!$, while for $n > 1$ the right side is not congruent to 0 ($n! - 1$ is odd). For $n > 1$ and odd, we have a contradiction.

## Problem 5

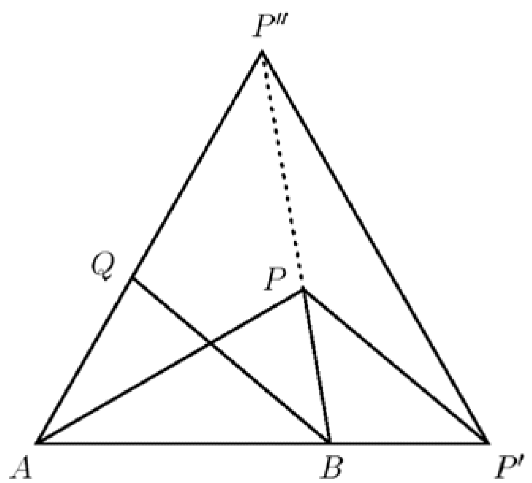
In a triangle $ABC$, let $AP$ bisect $\angle BAC$, with $P$ on $BC$, and let $BQ$ bisect $\angle ABC$, with $Q$ on $CA$.

It is known that $\angle BAC = 60^\circ$ and that $AB + BP = AQ + QB$.

What are the possible angles of triangle $ABC$?

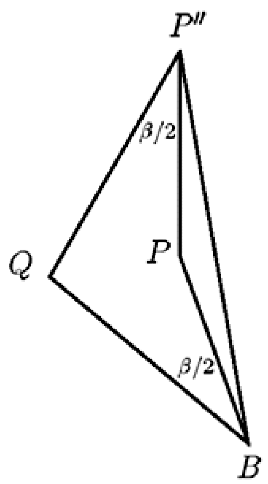
### Solution

Denote the angles of $ABC$ by $\alpha = 60^\circ$, $\beta$, and $\gamma$. Extend $AB$ to $P'$ so that $BP' = BP$, and construct $P''$ on $AQ$ so that $AP'' = AP'$. Then $BP'P$ is an isosceles triangle with base angle $\beta/2$. Since $AQ + QP'' = AB + BP' = AB + BP = AQ + QB$, it follows that $QP'' = QB$. Since $AP'P''$ is equilateral and $AP$ bisects the angle at $A$, we have $PP' = PP''$.




**Claim.** Points $B$, $P$, $P''$ are collinear, so $P''$ coincides with $C$.

*Proof*. Suppose to the contrary that $BPP''$ is a nondegenerate triangle. We have that $\angle PBQ = \angle PP'B = \angle PP''Q = \beta/2$. Thus the diagram appears as below, or else with $P$ is on the other side of $BP''$. In either case, the assumption that $BPP''$ is nondegenerate leads to $BP = PP'' = PP'$, thus to the conclusion that $BPP'$ is equilateral, and finally to the absurdity $\beta/2 = 60^\circ$ so $\alpha + \beta = 60^\circ + 120^\circ = 180^\circ$.




Thus points $B, P, P''$ are collinear, and $P'' = C$ as claimed.□

Since triangle $BCQ$ is isosceles, we have $120^\circ - \beta = \gamma = \beta/2$, so $\beta = 80^\circ$ and $\gamma = 40^\circ$. Thus $ABC$ is a 60-80-40 degree triangle.

## Problem 6

Let $a, b, c, d$ be integers with $a > b > c > d > 0$. Suppose that

$$a\,c + b\,d = (b + d + a - c)\,(b + d - a + c).$$

Prove that $a\,b + c\,d$ is not prime.

### Solution

- Solution 1

Suppose to the contrary that $a\,b + c\,d$ is prime. Note that

$$a\,b + c\,d = (a + d)\,c + (b - c)\,a = m \cdot \gcd\,(a + d, b - c)$$

for some positive integer $m$. By assumption, either $m = 1$ or $\gcd(a + d, b - c) = 1$. We consider these alternatives in turn.

*Case (i):* $m = 1$. Then

$$\begin{aligned} \gcd(a + d, b - c) &= a\,b + c\,d > a\,b + c\,d - (a - b + c + d) \\ &= (a + d)\,(c - 1) + (b - c)\,(a + 1) \\ &\geq \gcd(a + d, b - c), \end{aligned}$$

which is false.

*Case (ii):* $\gcd(a + d, b - c) = 1$. Substituting $a\,c + b\,d = (a + d)\,b - (b - c)\,a$ for the left-hand side of $a\,c + b\,d = (b + d + a - c)\,(b + d - a + c)$, we obtain

$$(a + d)\,(a - c - d) = (b - c)\,(b + c + d).$$

In view of this, there exists a positive integer $k$ such that

$$\begin{aligned} a-c-d &= k(b-c), \\ b+c+d &= k(a+d). \end{aligned}$$

Adding these equations, we obtain $a+b=k(a+b-c+d)$ and thus $k(c-d)=(k-1)(a+b)$. Recall that $a>b>c>d$. If $k=1$ then $c=d$, a contradiction. If $k \geq 2$ then

$$2 \geq \frac{k}{k-1} = \frac{a+b}{c-d} > 2,$$

a contradiction.

Since a contradiction is reached in both (i) and (ii), $a\,b+c\,d$ is not prime.

■ Solution 2

The equality $a\,c+b\,d=(b+d+a-c)(b+d-a+c)$ is equivalent to

$$a^2-a\,c+c^2=b^2+b\,d+d^2. \qquad (1)$$

Let $ABCD$ be the quadrilateral with $AB=a$, $BC=d$, $CD=b$, $AD=c$, $\angle BAD=60^\circ$, and $\angle BCD=120^\circ$. Such a quadrilateral exists in view of (1) and the Law of Cosines; the common value in (1) is $BD^2$. Let $\angle ABC=\alpha$, so that $\angle CDA=180^\circ-\alpha$. Applying the Law of Cosines to triangles $ABC$ and $ACD$ gives

$$a^2+d^2-2\,a\,d\cos\alpha = A\,C^2 = b^2+c^2+2\,b\,c\cos\alpha.$$

Hence $2\cos\alpha=(a^2+d^2-b^2-c^2)/(a\,d+b\,c)$, and

$$A\,C^2=a^2+d^2-a\,d\;\frac{a^2+d^2-b^2-c^2}{a\,d+b\,c} = \frac{(a\,b+c\,d)(a\,c+b\,d)}{a\,d+b\,c.}$$

Because $ABCD$ is cyclic, Ptolemy's Theorem gives

$$(A\,C\cdot B\,D)^2=(a\,b+c\,d)^2$$

It follows that

$$(a\,c+b\,d)(a^2-a\,c+c^2)=(a\,b+c\,d)(a\,d+b\,c). \qquad (2)$$

(*Note*. Straightforward algebra can also be used obtain (2) from (1).) Next observe that

$$a\,b+c\,d>a\,c+b\,d>a\,d+b\,c. \qquad (3)$$

The first inequality follows from $(a-d)(b-c)>0$, and the second from $(a-b)(c-d)>0$.

Now assume that $a\,b+c\,d$ is prime. It then follows from (3) that $a\,b+c\,d$ and $a\,c+b\,d$ are relatively prime. Hence, from (2), it must be true that $a\,c+b\,d$ divides $a\,d+b\,c$. However, this is impossible by (3). Thus $a\,b+c\,d$ must not be prime.

*Note*. Examples of 4-tuples $(a, b, c, d)$ that satisfy the given conditions are $(21, 18, 14, 1)$ and $(65, 50, 34, 11)$.